Def A continuous $(\mathcal{F}_t, P)$-<u>semimartingale</u> is a continuous process (X_t) which can be written $X_t = M_t + A_t$, where (M_t) - continuous local martingale.
(A_t) - continuous bounded variation adapted process.

Properties: 1) Since $\langle A, A\rangle_t = \langle M, A\rangle_t \equiv 0$, X has a finite quadratic variation $\langle X, X\rangle_t = \langle M, M\rangle_t$.

2) Can do polarization $\langle X, Y\rangle_t$, as before.

Def. Hardy space

$$H^2 := \{ M\text{-continuous martingale}, \ \|M\|^2_{H^2} := \sup_t E(M_t^2) < \infty \}.$$

For $M \in H^2$, by Martingale Convergence, $\exists M_\infty : M_t = E(M_\infty | \mathcal{F}_t)$,

$$E(M_\infty^2) = \sup(E(M_t^2)) = \lim_{t\to\infty} E(M_t^2).$$

$\mathbb{H}^2$ - Hilbert space, H^2 - closed subspace of martingales of the form $f_t = E(f_\infty | \mathcal{F}_t)$, $f_\infty \in L^2$. Indeed,

$M^n_\infty \to M_\infty$ in L^2 $\overset{\text{Martingale inequality}}{\Longrightarrow}$

$E\left(\left(\sup_t (M_t^n - M_t)\right)^2\right) \to 0 \Rightarrow$

One can select subsequence $M_t^{n_k} \to M_t$ uniformly in t

$H_0^2 := \{ M \in H^2 : M_0 \equiv 0 \}$ - another closed subspace.

Thm Let (M_t) be a continuous local martingale. The following are equivalent:

1) $M \in H^2$

2) a) $M_0 \in L^2$ and
b) $\lim_{t\to\infty} \langle M, M\rangle_t =: \langle M, M\rangle_\infty$, $E(\langle M, M\rangle_\infty) < \infty$.

Proof Let $T_n \nearrow \infty$ - stopping times; $(M_t^{T_n})$ - bounded martingale.

1) $\Rightarrow$ 2) Let $M \in H^2$. Then

$$E(M_0^2) = E\left(E(M_\infty \mid \mathcal{F}_0)\right)^2) \le E(M_\infty^2) < \infty$$

and

$$E(M_\infty^2) = \lim_{n\to\infty} E(M_{T_n}^2) = \lim_{n\to\infty} \overbrace{\left(E\left(M_{T_n}^2 - \langle M,M\rangle^{T_n}\right)\right)}^{=E(M_0^2)-\text{ martingale}} +$$

$$\lim_{n\to\infty} E\left(\langle M,M\rangle^{T_n}\right) = E(M_0^2) + E(\langle M,M\rangle_\infty).$$

2) ⇒ 1) By Fatou's lemma, $\forall t$.

$$E(M_t^2) \le \varliminf_{n\to\infty} E\left(M^2_{\min(T_n,t)}\right) = \varliminf_{n\to\infty}\left(E(M_0^2) + E(\langle M,M\rangle_{\min(T_n,t)})\right) =$$

$$E(M_0^2) + E(\langle M,M\rangle_t) \le E(M_0^2) + E(\langle M,M\rangle_\infty)$$

<u>Remark</u>. For $M \in H^2$, we have

$$\sup_t |M_t^2 - \langle M,M\rangle_t| \le \left(M_\infty^*\right)^2 + \langle M,M\rangle_\infty \in L^1$$

so $M_t^2 - \langle M,M\rangle_t$ — uniformly integrable martingale.

<u>Remark</u>. $B_t \notin H^2$, $B_t^T \in H^2 \iff E(T) < \infty$

(because $\langle B_t^T, B_t^T\rangle_\infty = T$).

Later, we'll need the following:

<u>Lemma</u>. Let (X_t) be a local martingale with continuous trajectories,

and $\forall \tau\ \exists c(\tau)$:

$$\langle X,X\rangle_t \le c(\tau)\int_0^t (X_s^2+1)\,ds \quad \text{if } t \le \tau.$$

Then (X_t) is a martingale.

<u>Proof</u>. Fix $M > 0$, $T^M := \inf\{t : |X_t| \ge M\}$.

$$Y_t^{T_M} = X_{\min(t,T^M)} \text{ — bounded martingale.}$$

$$f(t) := \langle Y, Y\rangle_t = \langle X,X\rangle_t^{T_M} = E(Y_t^2) = E\left(\left(X_t^{T_M}\right)^2\right)$$

So, by our assumption, $\forall t \le \tau$

$$f(t) \le c(\tau)\,t + c(\tau)\int_0^t f(s)\,ds$$

Let $t_0 = \min \{ t ; f(t) \geq \exp(2c(\tau)t) \}$.

Then
$$f(t_0) < c(\tau) t_0 + \frac{e^{2c(\tau)t_0} - 1}{2}. \quad \text{If } t_0 \leq \tau,$$

then $f(t_0) < e^{2c(\tau)t_0}$ — contradiction.

Thus, $\forall\ t \leq \tau$, $f(t) < e^{2c(\tau)t}$.

So $E(\langle X, X\rangle_t^{T_M}) \leq \exp(2c(\tau)t) \quad (t \leq \tau)$.

Take $M \to \infty$, to get that, by monotone convergence,
$$E(\langle X, X\rangle_t) \leq \exp(2c(\tau)t).$$

So, by previous thm, X_t^τ is an H^2-martingale

so for $\forall \tau$ if $s \leq t \leq \tau$ then
$$E(X_t \mid \mathcal{F}_s) = X_s. \quad \text{So } (X_t) \text{ is a martingale}$$
$$X_t^\tau = X_t, \quad X_s^\tau = X_s \quad \text{if } s \leq t \leq \tau$$